Exercise 1.5.2. Review the proof of Theorem 1.5.6, part (ii) showing that $\mathbb{R}$ is uncountable, and then find the flaw in the following erroneous proof that $\mathbb{Q}$ is uncountable:
Assume, for contradiction, that $\mathbb{Q}$ is countable. Thus we can write $\mathbb{Q} = \{r1, r2, r3, \dots\}$ and, as before, construct a nested sequence of closed intervals with $r_n \notin I_n$. Our construction implies $\cap_{n=1}^{\infty} I_n = \emptyset$ while NIP implies $\cap_{n=1}^{\infty} I_n \neq \emptyset$. This contradiction implies Q must therefore be uncountable.

(1) It may contain only one irrational number.
(2) NIP is for real intervals not rational.

Source: https://math.stackexchange.com/questions/1914901/false-proofs-claiming-that-mathbbq-is-uncountable

Exercise 1.5.4. (a) Show $(a, b) \sim R$ for any interval (a, b).
We know from the **Example 1.4.9.** that the function $f(x) = x/(x^2-1)$ takes the interval $(-1, 1)$ onto $\mathbb{R}$ in a 1–1 fashion. Then we map (a, b) onto $(-1, 1)$ by another bijective linear function $g(x) = 2x/(b-a) - (b+a)/(b-a)$.

(b) Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.
We know from the **Example 1.4.9.** that the function $f(x) = x/(x^2-1)$ takes the interval $(-1, 1)$ onto $\mathbb{R}$ in a 1–1 fashion. Then we map (a, ∞) onto $(-1, 1)$ by another bijective linear function $g(x) = 2x/(1-x)$.

(c) Using open intervals makes it more convenient to produce the required 1–1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1–1 onto function between the two sets.
$f : [0, 1) \to (0, 1)$ by $f(0) = 1/2$, $f(1/n) = 1/(n+1)$ for integer $n \geq 2$, and $f(x) = x$ otherwise.

Source: https://math.stackexchange.com/questions/1425492/explicit-bijection-between-0-1-and-0-1

Exercise 1.5.5. (a) Why is $A \sim A$ for every set A?
Trivial. By definition $f(x) = x$ will do the job.

(b) Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.
Bijection, so consider inverse mapping.

(c) For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an equivalence relation.
Assume f maps A to B and g maps B to C, $g(f(x))$ will work.

Exercise 1.5.6. (a) Give an example of a countable collection of disjoint open intervals.
$A_n = (n, n+1), n \in \mathbb{N}$

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.
DNE. Every collection of disjoint open intervals in $\mathbb{R}$ is countable because you can choose a rational number (by density theorem) in each of them and rationals are countable.

Exercise 1.5.7. Consider the open interval $(0, 1)$, and let S be the set of points in the open unit square; that is, $S = \{(x, y) : 0 < x, y < 1\}$.

(a) Find a 1–1 function that maps $(0, 1)$ into, but not necessarily onto, S. (This is easy.)
$f(x) = (x, x), x \in (0, 1)$

(b) Use the fact that every real number has a decimal expansion to produce a 1–1 function that maps S into $(0, 1)$. Discuss whether the formulated function is onto. (Keep in mind that any terminating decimal expansion such as $.235$ represents the same real number as $.234999\ldots$)

For any point with two coordinates $(0.d_1d_2\ldots, 0.e_1e_2\ldots)$, we map it to the real number $(0.d_1e_1d_2e_2\ldots)$. We restrict the choice of point in its simplest form so that $(0.2, 0.5)$ will be chosen for 0.25 instead of $(0.2999\ldots, 0.4999\ldots)$, which is equal to $(0.3, 0.5)$, corresponding to 0.35.

This function (mapping), however, is not onto. Consider $1/11 = 0.090909\ldots$, which by definition can be produced by a point $(0, 0.999\ldots)$, but this point can no be selected since it is equal to $(0, 1)$ and $(0, 1)$ yields 0.01. Therefore not point in the unit square can be used to map to $1/11$.

Exercise 1.5.8. Let B be a set of positive real numbers with the property that adding together any finite subset of elements from B always gives a sum of 2 or less. Show B must be finite or countable.

For each $n \in \mathbb{N}$, let

$$B_n = \left\{ b \in B \,\middle|\, b \geqslant \frac{2}{n} \right\} \subset B.$$

Of course, B_n can have no more than $n-1$ distinct elements; otherwise, the sum of n distinct elements of B_n would be grater than 2.

But

$$B = \bigcup_{n \in \mathbb{N}} B_n.$$

Since $\mathbb{N}$ is countable and each B_n is finite, B is countable.

Source: https://math.stackexchange.com/questions/2446630/showing-a-set-is-finite-or-countable

Exercise 1.5.10. (a) Let $C \subseteq [0, 1]$ be uncountable, show there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

Suppose that $C \cap [\frac{1}{n}, 1]$ is countable for all n. Then

$$C \cap [0,1] = C \cap \left(\{0\} \cup \bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 1]\right) = (C \cap \{0\}) \cup \bigcup_{n=1}^{\infty} (C \cap [\tfrac{1}{n}, 1])$$

would be countable too.

Source: https://math.stackexchange.com/questions/1452550/let-c-subseteq-0-1-be-uncountable-show-there-exists-a-in-0-1-such-tha

(b) Now let A be the set of all $a \in (0, 1)$ such that $C \subseteq [a, 1]$ is uncountable, and set $\alpha = supA$. Is $C \subseteq [0, 1]$ an uncountable set?

WTS: Suppose $C \subseteq [0, 1]$ is uncountable. Let $A = \{a \in (0, 1) \mid C \cap [a, 1] \text{ is uncountable } \}$, and $\alpha = \sup A$. Then $C \cap [\alpha, 1]$ is countable.

First, A is nonempty: for $n \in \mathbb{N}$ let $C_n = C \cap [\frac{1}{n}, 1]$. Some C_n must be uncountable, otherwise $C = \bigcup_n C_n$ is a countable union of countable sets and therefore countable. So for some n, $1/n \in A$.

Clearly $0 < \alpha \leq 1$.

If $\alpha = 1$ then of course the claim is true.
If $\alpha < 1$. Let (b_n) be a decreasing sequence in $(\alpha, 1)$ with $\alpha = \inf_n b_n$. By definition of A and α, for every n, $C \cap [b_n, 1]$ is countable, for otherwise $b_n \in A$ and $b_n \leq \alpha$. Thus

$$\begin{aligned} C \cap [\alpha, 1] &= C \cap \bigcup_n [b_n, 1] &(1) \\ &= \bigcup_n (C \cap [b_n, 1]) &(2) \end{aligned}$$

is a countable union of countable sets, so it's countable.

Source: https://math.stackexchange.com/questions/1639608/intersection-of-uncountable-sets

Exercise 1.5.11 (Schröder–Bernstein Theorem). Assume there exists a 1–1 function function $f : X \to Y$ and another 1–1 function $g : Y \to X$. Then there exists a 1–1, onto function $h : X \to Y$

and hence $X \sim Y$.

The strategy is to partition X and Y into components $X = A \cup A'$ and $Y = B \cup B'$ with $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$, in such a way that f maps A onto B and g maps B' onto A'.

(a) Explain how achieving this would lead to a proof that $X \sim Y$.
$f : A \to B$ is a 1–1, onto function;
$g : B' \to A'$ is a 1–1, onto function;
Then $h(x) = f(x)$ if $x \in A$ and $h(x) = g^{-1}(x)$ if $x \in A'$ is a $X \to Y$ 1–1, onto function and hence $X \sim Y$.

(b) Set $A_1 = X \setminus g(Y)$ (what happens if $A_1 = \emptyset$?) and inductively define a sequence of sets by letting $A_{n+1} = g(f(A_n))$. Show that $\{A_n : n \in \mathbb{N}\}$ is a pairwise disjoint collection of subsets of X, while $\{f(A_n) : n \in \mathbb{N}\}$ is a similar collection in Y.

For $k \geq 2$, since $A_k = g(f(A_{k-1})) \subseteq g(Y)$, A_k and A_1 are disjoint.

For $2 \leq m < n$, if there exists $a \in A_m \cap A_n$, then for some $a_{m-1} \in A_{m-1}$ and $a_{n-1} \in A_{n-1}$, $f(g(a_{m-1})) = a = f(g(a_{n-1}))$. Since both f and g are injective, here $a_{m-1} = a_{n-1}$. Hence $A_m \cap A_n \neq \emptyset$ implies $A_{m-1} \cap A_{n-1} \neq \emptyset$. By induction, we can conclude that $A_1 \cap A_{n-m+1} \neq \emptyset$, which is contradict with part 1. Therefore A_m and A_n are disjoint ($2 \leq m < n$).

Source: https://math.stackexchange.com/questions/1726578/understanding-a-proof-of-schröder-bernstein-theorem

(c) Let $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} f(A_n)$. Show that f maps A onto B.
Trivial because for every $b \in B$, $b = f(a_n)$ for some $a_n \in A_n \subseteq A$.

(d) Let $A' = X \setminus A$ and $B' = Y \setminus B$. Show g maps B' onto A'.
Suppose there is an element $a' \in A' \notin g(B')$. Since a' cannot be in A_1 there has to be an element $b \in f(A_n) \subset B$ s.t. $g(b) = a'$. Since $b \in f(A_n)$ we can write it as $f(a) = b$ and therefore $a' = g(f(a)) \in A_{n+1}$. But this is a contradiction to where a' lives.

Source: https://math.stackexchange.com/questions/1726578/understanding-a-proof-of-schröder-bernstein-theorem

Exercise 1.6.9. Using the various tools and techniques developed in the last two sections (including the exercises from Section 1.5), give a compelling argument showing that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$.

First note that that $\mathbb{R}$ can inject into $\mathcal{P}(\mathbb{Q})$ by mapping r to $\{q \in \mathbb{Q} \mid q < r\}$. Since $\mathbb{Q}$ is countable there is a bijection between $\mathcal{P}(\mathbb{Q})$ and $\mathcal{P}(\mathbb{N})$. So $\mathbb{R}$ injects into $\mathcal{P}(\mathbb{N})$.

Then note that we can map $x \in 2^{\mathbb{N}}$ to the continued fraction defined by the sequence x. Or to a point in $[0,1]$ defined by $\sum \frac{x(n)}{3^{n+1}}$, which we can show is injective in a somewhat easier proof.